

# INSTANTON VACUUM BEYOND CHIRAL LIMIT<sup>1</sup>

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## Abstract

In this talk it is discussed the derivation of low-frequencies part of quark determinant and partition function. As a first application, quark condensate is calculated beyond chiral limit with the account of  $O(m)$ ,  $O(\frac{1}{N_c})$ ,  $O(\frac{1}{N_c}m)$  and  $O(\frac{1}{N_c}m \ln m)$  corrections. It was demonstrated complete correspondence of the results to chiral perturbation theory.

## Introduction

Instanton vacuum model assume that QCD vacuum is filled not only by perturbative but also very strong non-perturbative fluctuations – instantons. This model provides a natural mechanism for the spontaneous breaking of chiral symmetry (SBCS) due to the delocalization of single-instanton quark zero modes in the instanton medium. The model is described by two main parameters – the average instanton size  $\rho \sim 0.3 fm$  and average inter-instanton distance  $R \sim 1 fm$ . These values was found phenomenologically [1] and theoretically [2] and was confirmed by lattice measurements [3, 4, 5, 6, 7]. On the base of this model was developed effective action approach [8, 9, 10], providing reliable method of the calculations of the observables in hadron physics at least in chiral limit.

On the other hand, chiral perturbation theory makes a theoretical framework incorporating the constraints on low-energy behavior of various observables based on the general principles of chiral symmetry and quantum field theory [11].

It is natural expect, that instanton vacuum model leads to the results compatible with chiral perturbation theory.

One of the most important quantities related with SBCS is the vacuum quark condensate  $\langle \bar{q}q \rangle$ , playing also important phenomenological role in various applications of QCD sum rule approach. Previous investigations [12] shows that beyond chiral limit and at small current quark mass  $m \sim few MeV$  these quantity receive large so called chiral log contribution  $\sim \frac{1}{N_c} m \ln m$  with fixed model independent coefficient. On the typical scale  $1 GeV$  it become leading correction since  $|\frac{1}{N_c} \ln m| \geq 1$ . It was shown, that this correction is due to pion loop contribution [12, 11].

So, to be consistent we have to calculate simultaneously all of the corrections of order  $m$ ,  $\frac{1}{N_c}$ ,  $\frac{1}{N_c} \ln m$  in order to find quark condensate beyond chiral limit.

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In our previous papers [13, 14] on the base of low-frequencies part of light quark determinant  $\text{Det}_{low}$ , obtained in [15, 8, 16], was derived effective action. In this framework was investigated current quark mass  $m$  dependence of the quark condensate, but without meson loop contribution [14].

In the present work we refine the derivation of the low-frequencies part of light quark determinant  $\text{Det}_{low}$ . The following averaging of  $\text{Det}_{low}$  over instanton collective coordinates is done independently over each instanton thanks to small packing parameter  $\pi(\frac{\rho}{R})^4 \sim 0.1$  and also by introducing constituent quarks degree of freedoms  $\psi$ . This procedure leads to the light quarks partition function  $Z[m]$ . We apply bosonisation procedure to  $Z[m]$ , which is exact one for our case  $N_f = 2$  and calculate partition function  $Z[m]$  with account of meson loops. This one provide us the quark condensate with desired  $O(m)$ ,  $O(\frac{1}{N_c})$ ,  $O(\frac{1}{N_c}m \ln m)$  corrections.

## Low-frequencies part of light quark determinant

The main assumption of previous works [8, 9, 10] (see also review [16]) was that at very small  $m$  the quark propagator in the single instanton field  $A_i$  can be approximated as:

$$S_I(m \sim 0) \approx \frac{1}{i\hat{\partial}} + \frac{|\Phi_{0I} \rangle \langle \Phi_{0I}|}{im} \quad (1)$$

It gives proper value for the  $\langle \Phi_{0I} | S_I(m \sim 0) | \Phi_{0I} \rangle = \frac{1}{im}$ , but in  $S_I(m \sim 0) | \Phi_{0I} \rangle = \frac{|\Phi_{0I} \rangle}{im} + \frac{1}{i\hat{\partial}} | \Phi_{0I} \rangle$  second extra term has a wrong chiral properties. We may neglect by this one only for the  $m \sim 0$ .

At the present case of non-small  $m$  we assume:

$$S_I \approx S_0 + S_0 i\hat{\partial} \frac{|\Phi_{0I} \rangle \langle \Phi_{0I}|}{c_I} i\hat{\partial} S_0, \quad S_0 = \frac{1}{i\hat{\partial} + im} \quad (2)$$

where

$$c_I = - \langle \Phi_{0I} | i\hat{\partial} S_0 i\hat{\partial} | \Phi_{0I} \rangle = im \langle \Phi_{0I} | S_0 i\hat{\partial} | \Phi_{0I} \rangle \quad (3)$$

The matrix element  $\langle \Phi_{0I} | S_I | \Phi_{0I} \rangle = \frac{1}{im}$ , more over

$$S_I | \Phi_{0I} \rangle = \frac{1}{im} | \Phi_{0I} \rangle, \quad \langle \Phi_{0I} | S_I = \langle \Phi_{0I} | \frac{1}{im} \quad (4)$$

as it must be.

In the field of instanton ensemble, represented by  $A = \sum_I A_I$ , full quark propagator, expanded with respect to a single instanton, and with account Eq. (2) is:

$$\begin{aligned} S &= S_0 + \sum_I (S_I - S_0) + \sum_{I \neq J} (S_I - S_0) S_0^{-1} (S_J - S_0) \\ &+ \sum_{I \neq J, J \neq K} (S_I - S_0) S_0^{-1} (S_J - S_0) S_0^{-1} (S_K - S_0) + \dots \\ &= S_0 + \sum_{I,J} S_0 i\hat{\partial} | \Phi_{0I} \rangle \left( \frac{1}{C} + \frac{1}{C} T \frac{1}{C} + \dots \right)_{IJ} \langle \Phi_{0J} | i\hat{\partial} S_0 \\ &= S_0 + \sum_{I,J} S_0 i\hat{\partial} | \Phi_{0I} \rangle \left( \frac{1}{C - T} \right)_{IJ} \langle \Phi_{0J} | i\hat{\partial} S_0 \end{aligned} \quad (5)$$

where

$$\begin{aligned} C_{IJ} &= \delta_{IJ} c_I = -\delta_{IJ} < \Phi_{0I} | i\hat{\partial} S_0 i\hat{\partial} | \Phi_{0I} >, \\ (C - T)_{IJ} &= - < \Phi_{0I} | i\hat{\partial} S_0 i\hat{\partial} | \Phi_{0J} > \end{aligned} \quad (6)$$

We are calculating  $\text{Det}_{low}$  using the formula:

$$\ln \text{Det}_{low} = \text{Tr} \int_{M_1}^m idm' (\tilde{S}(m') - \tilde{S}_0(m')) \quad (7)$$

Within zero-mode assumption (Eq. (2)) the trace is restricted to the subspace of instantons:

$$\text{Tr}(S - S_0) = - \sum_{I,J} < \Phi_{0,J} | i\hat{\partial} S_0^2 i\hat{\partial} | \Phi_{0,I} > < \Phi_{0,I} | (\frac{1}{i\hat{\partial} S_0 i\hat{\partial}}) | \Phi_{0,J} > \quad (8)$$

Introducing now the matrix

$$B(m)_{IJ} = < \Phi_{0,I} | i\hat{\partial} S_0 i\hat{\partial} | \Phi_{0,J} > \quad (9)$$

it is easy to show that

$$\begin{aligned} \ln \text{Det}_{low} &= \text{Tr} \int_{M_1}^m idm' (S(m') - S_0(m')) = \sum_I \int_{B(M_1)}^{B(m)} (dB(m') \frac{1}{B(m')})_{II} \\ &= \text{Tr} \ln \frac{B(m)}{B(M_1)} = \ln \det B(m) - \ln \det B(M_1) \end{aligned} \quad (10)$$

which is desired answer. The determinant  $\det B(m)$  from Eq. (10) is the extension of the Lee-Bardeen result [15] for the non-small values of current quark mass  $m$ .

## Light quark effective action beyond chiral limit

Averaged  $\text{Det}_{low}$  leads to the partition function  $Z[m]$ , which for  $N_f = 2$  has the form:

$$\begin{aligned} Z[m] &= \int d\lambda_+ d\lambda_- D\psi D\psi^\dagger \exp \left[ \int d^4x \sum_{f=1}^2 \psi_f^\dagger (i\hat{\partial} + im_f) \psi_f \right. \\ &\quad \left. + \lambda_+ Y_2^+ + \lambda_- Y_2^- + N_+ \ln \frac{K}{\lambda_+} + N_- \ln \frac{K}{\lambda_-} \right], \end{aligned} \quad (11)$$

here  $\lambda_\pm$  are dynamical couplings ( $K$  is unessential constant, which provide under-logarithm expression dimensionless) [9, 13, 14]. Values of them are defined by saddle-point calculations.  $Y_2^\pm$  are t'Hooft type interaction terms [10]:

$$\begin{aligned} Y_2^\pm &= \frac{1}{N_c^2 - 1} \int d^4x \left[ \left(1 - \frac{1}{2N_c}\right) \det iJ^\pm(\rho, x) + \frac{1}{8N_c} \det iJ_{\mu\nu}^\pm(x) \right] \\ J_{fg}^\pm(x) &= \int \frac{d^4k_f d^4l_g}{(2\pi)^8} \exp i(k_f - l_g)x q_f^+(k_f) \frac{1 \pm \gamma_5}{2} q_g(l_g) \\ J_{\mu\nu, fg}^\pm(x) &= \int \frac{d^4k_f d^4l_g}{(2\pi)^8} \exp i(k_f - l_g)x q_f^+(k_f) \frac{1 \pm \gamma_5}{2} \sigma_{\mu\nu} q_g(l_g) \end{aligned} \quad (12)$$

where  $q(k) = 2\pi\rho F(k)\psi(k)$ . The form-factor  $F(k)$  is due to zero-modes and has explicit form  $F(k) = -\frac{d}{dt}[I_0(t)K_0(t) - I_1(t)K_1(t)]_{t=\frac{|k|\rho}{2}}$ . In the following we will neglect by  $J_{\mu\nu,fg}^\pm(x)$  interaction term, since it give a  $O(\frac{1}{N_c^2})$  contribution to the quark condensate. Since  $q(x) = \int \frac{d^4k}{(2\pi)^4} \exp(ikx) q(k)$ ,  $J_{fg}^\pm(x) = q_f^\pm(x) \frac{1 \pm \gamma_5}{2} q_g(x)$ ,

$$\begin{aligned} & \det \frac{iJ^+(x)}{g} + \det \frac{iJ^-(x)}{g} \\ &= \frac{1}{8g^2} (-(q^+(x)q(x))^2 - (q^+(x)i\gamma_5\vec{\tau}q(x))^2 + (q^+(x)\vec{\tau}q(x))^2 + (q^+(x)i\gamma_5q(x))^2). \end{aligned} \quad (13)$$

Here color factor  $g^2 = \frac{(N_c^2-1)2N_c}{(2N_c-1)}$ .

In the following we will take equal number of instantons and antiinstantons  $N_+ = N_- = N/2$  and corresponding couplings  $\lambda_\pm = \lambda$ .

Now it is natural to bosonize quark-quark interaction terms (13) by introducing meson fields. For  $N_f = 2$  case it is exact procedure. We have to take into account the changes of  $q$  and  $q^\dagger$  under the  $SU(2)$  chiral transformations:

$$\delta q = i\gamma_5\vec{\tau}\vec{\alpha}q, \quad \delta q^+ = q^+i\gamma_5\vec{\tau}\vec{\alpha}$$

to introduce appropriate meson fields, changing under  $SU(2)$  chiral transformations as:

$$\delta\sigma = 2\vec{\alpha}\vec{\phi}, \quad \delta\vec{\phi} = -2\vec{\alpha}\sigma, \quad \delta\eta = -2\vec{\alpha}\vec{\sigma}, \quad \delta\vec{\sigma} = 2\eta\vec{\alpha}.$$

Then  $\delta q^+(\sigma + i\gamma_5\vec{\tau}\vec{\phi})q = 0$ ,  $\delta q^+(\vec{\tau}\vec{\sigma} + i\gamma_5\eta)q = 0$  means that these combinations of fields are chiral invariant<sup>2</sup>. So, the interaction term has an exact bosonized representation:

$$\begin{aligned} & \int d^4x \exp[\lambda(\det \frac{iJ^+}{g} + \det \frac{iJ^-}{g})] \\ &= \int D\sigma D\vec{\phi} D\eta D\vec{\sigma} \exp \int d^4x [\frac{\lambda^{0.5}}{2g} q^+ i(\sigma + i\gamma_5\vec{\tau}\vec{\phi} + i\vec{\tau}\vec{\sigma} + \gamma_5\eta)q - \frac{1}{2}(\sigma^2 + \vec{\phi}^2 + \vec{\sigma}^2 + \eta^2)] \end{aligned} \quad (14)$$

Then the partition function is

$$\begin{aligned} Z[m] &= \int d\lambda D\sigma D\vec{\phi} D\eta D\vec{\sigma} \exp[N \ln \frac{K}{\lambda} - N \\ & - \frac{1}{2} \int d^4x (\sigma^2 + \vec{\phi}^2 + \vec{\sigma}^2 + \eta^2) + \text{Tr} \ln \frac{\hat{p} + im + i\frac{\lambda^{0.5}}{2g} (2\pi\rho)^2 F(\sigma + i\gamma_5\vec{\tau}\vec{\phi} + i\vec{\tau}\vec{\sigma} + \gamma_5\eta)F}{\hat{p} + im}] \end{aligned} \quad (15)$$

( $\text{Tr}(\dots)$  means here  $\text{tr}_{\gamma,c,f} \int d^4x \langle x | (\dots) | x \rangle$ , where  $\text{tr}_{\gamma,c,f}$  is the trace over Dirac, color, and flavor indexes.) In the following we assume  $m_u = m_d = m$ . Then common saddle point on  $\lambda$ ,  $\sigma$  ( $= \text{const}$ ) ( $\text{others} = 0$ ) is defined by Eqs.  $\frac{\partial V[m,\lambda,\sigma]}{\partial \lambda} = \frac{\partial V[m,\lambda,\sigma]}{\partial \sigma} = 0$ , where the potential

$$V[m, \lambda, \sigma] = -N \ln \frac{K}{\lambda} + N + \frac{1}{2} V \sigma^2 - \text{Tr} \ln \frac{\hat{p} + i(m + M(\lambda, \sigma)F^2(p))}{\hat{p} + im} \quad (16)$$

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<sup>2</sup>Certainly, quark-quark interaction term Eq. (13) is non-invariant over  $U(1)$  axial transformations, as it must be.

and we defined  $M(\lambda, \sigma) = \frac{\lambda^{0.5}}{2g}(2\pi\rho)^2\sigma$ . Then the common saddle-point on  $\lambda$  and  $\sigma$  is given by Eqs.:

$$N = \frac{1}{2} \text{Tr} \frac{iM(\lambda, \sigma)F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma)F^2(p))} = \frac{1}{2} V\sigma^2. \quad (17)$$

The solutions of this Eqs. are  $\lambda_0$  and  $\sigma_0 = (2\frac{N}{V})^{1/2} = 2^{1/2}R^{-2}$ . It is clear that  $M_0 = M(\lambda_0, \sigma_0)$  has a meaning of dynamical quark mass, which is defined by this Eqs.. At typical values  $R^{-1} = 200 \text{ MeV}$ ,  $\rho^{-1} = 600 \text{ MeV}$  we have  $\sigma_0^2 = 2(200 \text{ MeV})^4$ , and in chiral limit  $m = 0$   $M_0 \rightarrow M_{00} = 358 \text{ MeV}$ ,  $\lambda_{00} \approx M_{00}^2$ . It is clear that due to saddle-point equation (17)  $M_0$  (and  $\lambda_0$ ) become the function of the current mass  $m$ . This dependence was investigated in [14].

## Vacuum with account of quantum corrections

The account of the quantum fluctuations around saddle-points  $\sigma_0, \lambda_0$  will change the potential  $V[m, \lambda, \sigma]$  to  $V_{eff}[m, \lambda, \sigma]$  (it is clear that the difference between these two potentials is order of  $1/N_c$ ). Then, the partition function is given by Eq.

$$Z[m] = \int d\lambda \exp(-V_{eff}[m, \lambda, \sigma]) \quad (18)$$

There is important difference between this instanton generated partition function  $Z[m]$  and traditional  $NJL$ -type models – we have to integrate over the coupling  $\lambda$  here. As was mentioned before, this integration on  $\lambda$  by saddle-point method leads to exact answer. This saddle-point is defined by Eq.:

$$\frac{dV_{eff}[m, \lambda, \sigma]}{d\lambda} = 0 \quad (19)$$

which leads to the  $\lambda$  as a function of  $\sigma$ , i.e.  $\lambda = \lambda(\sigma)$ .

Then, the vacuum is the minimum of the effective potential  $V_{eff}[m, \sigma]$ , which is given by a solution of the equation

$$\frac{dV_{eff}[m, \sigma, \lambda(\sigma)]}{d\sigma} = \frac{\partial V_{eff}[m, \sigma, \lambda(\sigma)]}{\partial \sigma} = 0. \quad (20)$$

where it was used Eq. (19).

We denote a fluctuations as a primed fields  $\Phi'_i$ . The action and corresponding  $V_{eff}$  now has a form:

$$S[m, \lambda, \sigma, \Phi'] = S_0[m, \lambda, \sigma] + S_V[m, \lambda, \sigma, \Phi'], \quad (21)$$

$$S_0[m, \lambda, \sigma] = V[m, \lambda, \sigma] = \frac{1}{2} V\sigma^2 - \text{Tr} \ln \frac{\hat{p} + i(m + M(\lambda, \sigma)F^2)}{\hat{p} + im} - N \ln \frac{K}{\lambda} + N$$

$$S_V[m, \lambda, \sigma, \Phi'] = \int d^4x \frac{1}{2} (\sigma'^2 + \vec{\phi}^{\prime 2} + \vec{\sigma}'^2 + \eta'^2) - \frac{1}{2\sigma^2} \text{Tr} \left[ \frac{iM(\lambda, \sigma)F^2}{\hat{p} + i(m + M(\lambda, \sigma)F^2)} (\sigma' + i\gamma_5 \vec{\tau} \vec{\phi}' + i\vec{\tau} \vec{\sigma}' + \gamma_5 \eta') \right]^2,$$

and

$$V_{eff}[m, \lambda, \sigma] = S_0[m, \lambda, \sigma] + V_{eff}^{mes}[m, \lambda, \sigma] \quad (22)$$

Here second term in Eq. (22) is explicitly represented by

$$V_{eff}^{mes}[m, \lambda, \sigma] = \frac{1}{2} \text{Tr} \ln \frac{\delta^2 S_V[m, \lambda, \sigma, \Phi']}{\delta \Phi'_i(x) \delta \Phi'_j(y)} = \frac{V}{2} \sum_i \int \frac{d^4 q}{(2\pi)^4} \ln [1 - \text{tr} \frac{1}{\sigma^2} \int \frac{d^4 p}{(2\pi)^4} \\ \times \frac{M(\lambda, \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma) F^2(p))} \Gamma_i \frac{M(\lambda, \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda, \sigma) F^2(p + q))} \Gamma_i], \quad (23)$$

where the factors  $\Gamma_i = (1, i\gamma_5 \vec{\tau}, i\vec{\tau}, \gamma_5)$  and the sum on  $i$  is counted all corresponding meson fluctuations  $\sigma', \vec{\phi}', \vec{\sigma}', \eta'$ .  $\text{tr}$  here means the trace over flavor, color and Dirac indexes. Integrals in Eq. (23) are completely convergent one due to the presence of the form-factors  $F$ .

Certainly the quantum fluctuations contribution will move the coupling  $\lambda$  from  $\lambda_0$  to  $\lambda_0 + \lambda_1$  and  $\sigma$  as  $\sigma_0 \rightarrow \sigma_0 + \sigma_1$ , where  $\frac{\lambda_1}{\lambda_0}$  and  $\frac{\sigma_1}{\sigma_0}$  are of order  $1/N_c$ .

First, consider Eq. (19):

$$\lambda \frac{dV_{eff}[m, \lambda, \sigma]}{d\lambda} = N - \frac{1}{2} \text{Tr} \frac{iM(\lambda, \sigma) F^2}{\hat{p} + i(m + M(\lambda, \sigma) F^2)} + \frac{V}{2} \sum_i \int \frac{d^4 q}{(2\pi)^4} \\ \times [\sigma^2 - \text{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{M(\lambda, \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma) F^2(p))} \Gamma_i \frac{M(\lambda, \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda, \sigma) F^2(p + q))} \Gamma_i]^{-1} \\ \times [-\text{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{M(\lambda, \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma) F^2(p))} \Gamma_i \frac{M(\lambda, \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda, \sigma) F^2(p + q))} \Gamma_i \\ + i \text{tr} \int \frac{d^4 p}{(2\pi)^4} \left( \frac{M(\lambda, \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma) F^2(p))} \right)^2 \Gamma_i \frac{M(\lambda, \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda, \sigma) F^2(p + q))} \Gamma_i] = 0$$

From this saddle-point Eq. we get  $\lambda = \lambda(\sigma)$ .

From vacuum Eq. (20) we in similar manner arrive to:

$$\sigma \frac{\partial V_{eff}[m, \sigma, \lambda(\sigma)]}{\partial \sigma} = V \sigma^2 - \text{Tr} \frac{iM(\lambda(\sigma), \sigma) F^2}{\hat{p} + i(m + M(\lambda(\sigma), \sigma) F^2)} + \frac{V}{2} \sum_i \int \frac{d^4 q}{(2\pi)^4} \\ \times [\sigma^2 - \text{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{M(\lambda(\sigma), \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda(\sigma), \sigma) F^2(p))} \Gamma_i \frac{M(\lambda(\sigma), \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda(\sigma), \sigma) F^2(p + q))} \Gamma_i]^{-1} \\ \times [2i \text{tr} \int \frac{d^4 p}{(2\pi)^4} \left( \frac{M(\lambda(\sigma), \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda(\sigma), \sigma) F^2(p))} \right)^2 \Gamma_i \frac{M(\lambda(\sigma), \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda(\sigma), \sigma) F^2(p + q))} \Gamma_i] = 0$$

Since we are believing to  $\frac{1}{N_c}$  expansion, it is natural inside quantum fluctuations contribution (under the integrals over  $q$ ) to take  $\sigma = \sigma_0$ ,  $M(\lambda(\sigma), \sigma) = M_0$ .

To simplify the expressions introduce vertices  $V_{2i}(q)$ ,  $V_{3i}(q)$  and meson propagators  $\Pi_i(q)$ , which are defined as:

$$V_{2i}(q) = \text{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{M_0(p)}{\hat{p} + i\mu_0(p)} \Gamma_i \frac{M_0(p + q)}{\hat{p} + \hat{q} + i\mu_0(p + q)} \Gamma_i \quad (26)$$

$$V_{3i}(q) = \text{tr} \int \frac{d^4 p}{(2\pi)^4} \left( \frac{M_0(p)}{\hat{p} + i\mu_0(p)} \right)^2 \Gamma_i \frac{M_0(p + q)}{\hat{p} + \hat{q} + i\mu_0(p + q)} \Gamma_i \quad (27)$$

$$\Pi_i^{-1}(q) = \frac{2}{R^4} - V_{2i}(q). \quad (28)$$

Here  $M_0(p) = M_0 F^2(p)$ ,  $\mu_0(p) = m + M_0(p)$  and was taken into account that  $\sigma_0^2 = 2R^{-4}$ .

From Eqs. (24) and (25) we have

$$\frac{M_1}{M_0} \left[ \frac{2}{R^4} + \frac{1}{V} \text{Tr} \left( \frac{M_0(p)}{\hat{p} + i\mu_0(p)} \right)^2 \right] = \sum_i \int \frac{d^4 q}{(2\pi)^4} (iV_i^3(q) - V_{2i}(q)) \Pi_i(q) \quad (29)$$

$$\frac{\sigma_1}{\sigma_0} = -\frac{R^4}{4} \sum_i \int \frac{d^4 q}{(2\pi)^4} V_{2i}(q) \Pi_i(q) \quad (30)$$

The vertices  $V_{2i}(q)$ ,  $V_{3i}(q)$  and the meson propagators  $\Pi_i(q)$  are well defined functions, providing well convergence of the integrals in Eqs. (29), (30).

It is of special attention to **the contribution of pion fluctuations  $\vec{\phi}'$  at small pion momentum  $q$** . We shall demonstrate that this contribution leads to the famous chiral log term with model independent coefficient in the correspondence with previous calculations in NJL-model [18].

Pion inverse propagator of  $\Pi_{\vec{\phi}'}^{-1}(q)$  at small  $q \sim m_\pi$  is:  $\Pi_{\vec{\phi}'}^{-1}(q) = f_{kin}^2(m_\pi^2 + q^2)$ . At lowest order on  $m$ ,  $f_{kin,m=0} \approx f_\pi = 93 \text{ MeV}$ ,  $m_\pi^2 \sim m$ .

The vertices in the right side of Eq. (29) at  $q = 0$  and in chiral limit are:

$$V_{2\vec{\phi}',m=0}(0) = \frac{2}{R^4}, \quad iV_{3\vec{\phi}',m=0}(0) - V_{2\vec{\phi}',m=0}(0) = 8N_c \int \frac{d^4 p}{(2\pi)^4} \frac{p^2 M_0^2(p)}{(p^2 + M_0^2(p))^2} \quad (31)$$

We see that the factor in the left side of Eq. (29) in the chiral limit is equal to:

$$\text{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{i\hat{p}M_0(p)}{(\hat{p} + iM_0(p))^2} = -2(iV_{3\vec{\phi}',m=0}(0) - V_{2\vec{\phi}',m=0}(0)) \quad (32)$$

Collecting all the factors we get small  $q \leq \kappa$  contribution of pion fluctuations  $\vec{\phi}'$ :

$$\begin{aligned} \frac{\sigma_1}{\sigma_0} \Big|_{\vec{\phi}', \text{small } q} &= \frac{M_1}{M_0} \Big|_{\vec{\phi}', \text{small } q} = -\frac{3}{2f_\pi^2} \int_0^\kappa \frac{d^4 q}{(2\pi)^4} \frac{1}{m_\pi^2 + q^2} \\ &= -\frac{3}{32\pi^2 f_\pi^2} \int_0^{\kappa^2} q^2 dq^2 \frac{1}{f_\pi^2(m_\pi^2 + q^2)} = -\frac{3}{32\pi^2 f_\pi^2} (\kappa^2 + m_\pi^2 \ln \frac{m_\pi^2}{\kappa^2 + m_\pi^2}) \end{aligned} \quad (33)$$

Here we put  $m = 0$  everywhere except  $m_\pi$ . We see that the coefficient in the front of of  $m_\pi^2 \ln m_\pi^2$  is a model independent as it must be.

Quick estimate, assuming  $\kappa = \rho^{-1}$ , gives

$$\frac{\sigma_1}{\sigma_0} \Big|_{\vec{\phi}'} \approx \frac{M_1}{M_0} \Big|_{\vec{\phi}'} \approx -\frac{3}{32\pi^2 f_\pi^2 \rho^2} (1 + m_\pi^2 \rho^2 \ln m_\pi^2 \rho^2) \approx -0.4(1 + 0.054 \ln 0.054) \quad (34)$$

So, we expect that pion loops is provided not only non-analytical  $\frac{1}{N_c} m \ln m$  term but also very large contribution to  $\frac{1}{N_c}$  term.

This one dictate the strategy of the following calculations of  $\sigma_1$  and  $M_1$ :

1. we have to extract analytically  $\frac{1}{N_c} m \ln m$  term from pion loops;
2. rest part of  $\sigma_1$  and  $M_1$  can be calculated numerically and expanded over  $m$ , paying special attention to the pion loops and keeping  $\frac{1}{N_c}$  and  $\frac{1}{N_c} m$  terms.

For actual numerical calculations we are using simplified version of the form-factor  $F(p)$  from [17] (with corrected high momentum dependence):

$$F(p < 2GeV) = \frac{L^2}{L^2 + p^2}, \quad F(p > 2GeV) = \frac{1.414}{p^3} \quad (35)$$

where  $L \approx \frac{\sqrt{2}}{\bar{\rho}} = 848 MeV$ .

At  $N_c = 3$  semi-numerical calculations of  $M_1$  and  $\sigma_1$  lead to:

$$\frac{M_1}{M_0} = -0.662 - 4.64m - 4.01m \ln m \quad (36)$$

$$\frac{\sigma_1}{\sigma_0} = -0.523 - 4.26m - 4.00m \ln m \quad (37)$$

Here  $m$  is given in  $GeV$ . Certainly, in (36) the  $m \ln m$  term is completely correspond to Eq. (33).  $\frac{M_1}{M_0}$  is  $-66\%$  in chiral limit and reach its maximum  $\sim -20\%$  at  $m \sim 0.115 GeV$ .

The relative shift of the vacuum  $\frac{\sigma_1}{\sigma_0}$  is  $-52\%$  at the chiral limit and reach its maximum  $\sim -2\%$  at  $m \sim 0.125 GeV$ .

The main contribution to both quantities  $\frac{M_1}{M_0}$  and  $\frac{\sigma_1}{\sigma_0}$  come from pion loops. Other mesons give the contribution  $\sim 10\%$  to  $O(\frac{1}{N_c})$  and  $O(\frac{1}{N_c}m)$  terms.

## Quark condensate

We have to calculate quark condensate beyond chiral limit taking into account  $O(m)$ ,  $O(\frac{1}{N_c})$ ,  $O(\frac{1}{N_c}m)$  and  $O(\frac{1}{N_c}m \ln m)$  terms. Quark condensate is extracted from the partition function:

$$\begin{aligned} \langle \bar{q}q \rangle &= \frac{1}{2V} \frac{dV_{eff}[m, \lambda, \sigma]}{dm} = \frac{1}{2V} \frac{\partial(V[m, \lambda, \sigma] + V_{eff}^{mes}[m, \lambda_0, \sigma_0])}{\partial m} \\ &= -\frac{1}{2V} \text{Tr} \left( \frac{i}{\hat{p} + i\mu(p)} - \frac{i}{\hat{p} + im} \right) + \frac{1}{2V} \frac{\partial V_{eff}^{mes}[m, \lambda_0, \sigma_0]}{\partial m} \end{aligned} \quad (38)$$

here  $\lambda = \lambda_0 + \lambda_1$ ,  $\sigma = \sigma_0 + \sigma_1$ ,  $M = M_0 + M_1$ ,  $\mu(p) = m + MF^2(p)$ . First term of Eq. (38) is

$$\begin{aligned} &-\frac{1}{2V} \text{Tr} \left( \frac{i}{\hat{p} + i\mu(p)} - \frac{i}{\hat{p} + im} \right) \\ &= -4N_c \int \frac{d^4p}{(2\pi)^4} \left( \frac{\mu_0(p)}{p^2 + \mu_0^2(p)} - \frac{m}{p^2 + m^2} + \frac{M_1}{M_0} \frac{M_0(p)(p^2 - \mu_0^2(p))}{(p^2 + \mu_0^2(p))^2} \right) \end{aligned} \quad (39)$$

Second term of Eq. (38) – meson loops contribution to the condensate is

$$\begin{aligned} \frac{1}{2V} \frac{\partial V_{eff}^{mes}[m, \lambda_0, \sigma_0]}{\partial m} &= \frac{i}{2} \sum_i \int \frac{d^4q}{(2\pi)^4} \left( \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{M_0(p)}{(\hat{p} + i\mu_0(p))^2} \Gamma_i \frac{M_0(p+q)}{\hat{p} + \hat{q} + i\mu_0(p+q)} \Gamma_i \right) \\ &\times \left( \frac{2N}{V} - \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{M_0(p)}{\hat{p} + i\mu_0(p)} \Gamma_i \frac{M_0(p+q)}{\hat{p} + \hat{q} + i\mu_0(p+q)} \Gamma_i \right)^{-1} \end{aligned} \quad (40)$$



At  $m = 0$  and without meson loops the condensate is

$$\langle \bar{q}q \rangle_{00} = -4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M_{00}(p)}{p^2 + M_{00}^2(p)} \quad (41)$$

Here  $M_{00} \equiv M_{0,m=0}$ .

Let us to consider now the contribution of pion fluctuations  $\vec{\phi}'$  to the quark condensate at small  $q$ . First we consider:

$$\frac{1}{2V} \frac{\partial V_{eff}^{\vec{\phi}', small q}[m, \lambda_0, \sigma_0]}{\partial m} = 12N_c \int \frac{d^4p}{(2\pi)^4} \frac{M_0^2(p) \mu_0(p)}{(p^2 + \mu_0^2(p))^2} \int_0^\kappa \frac{d^4q}{(2\pi)^4 f_{kin}^2 (m_\pi^2 + q^2)} \quad (42)$$

We keep  $m$  only in  $m_\pi^2$ . Then at  $m = 0$   $\mu_0(p) \Rightarrow M_0(p) \Rightarrow M_{00}(p)$ ,  $f_{kin} \Rightarrow f_\pi$  and we have

$$\langle \bar{q}q \rangle = \langle \bar{q}q \rangle_{00} - \frac{M_1}{M_0} 4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M_{00}(p)(p^2 - M_{00}^2(p))}{(p^2 + M_{00}^2(p))^2} \quad (43)$$

$$+ 12N_c \int \frac{d^4p}{(2\pi)^4} \frac{M_{00}^3(p)}{(p^2 + M_{00}^2(p))^2} \int_0^\kappa \frac{d^4q}{(2\pi)^4} \frac{1}{f_\pi^2 (m_\pi^2 + q^2)} \\ = \langle \bar{q}q \rangle_{00} \left( 1 - \frac{3}{2} \int_0^\kappa \frac{d^4q}{(2\pi)^4} \frac{1}{f_\pi^2 (m_\pi^2 + q^2)} \right) \quad (44)$$

Eq. (33) for  $\frac{M_1}{M_0}$  was applied here. We see that Eq. (44) is in the full correspondence with [11, 12].

Detailed numerical calculations lead to the semi-analytical formula for the quark condensate including all  $O(m)$ ,  $O(\frac{1}{N_c})$  and  $O(\frac{1}{N_c} m \ln m)$ -corrections:

$$\langle \bar{q}q \rangle = \langle \bar{q}q \rangle_{m=0} (1 - 18.53 m - 7.72 m \ln m) \quad (45)$$

Here  $\langle \bar{q}q \rangle_{m=0} = 0.52 \langle \bar{q}q \rangle_{00}$ . Certainly, the  $m \ln m$  term in Eq.(45) is in full correspondence with Eq. (44), as it must be.  $\langle \bar{q}q \rangle / \langle \bar{q}q \rangle_{m=0}$  is a rising function of  $m$  until  $m \sim 0.04 \text{ GeV}$  and is a falling one in the region  $m > 0.04 \text{ GeV}$ .

The main contribution to  $O(\frac{1}{N_c})$ ,  $O(\frac{1}{N_c} m)$  and  $O(\frac{1}{N_c} m \ln m)$  terms in  $\frac{\langle \bar{q}q \rangle}{\langle \bar{q}q \rangle_{00}}$  is due to pion loops. Other mesons give the contribution  $\sim \text{few} \%$  to  $O(\frac{1}{N_c})$  and  $O(\frac{1}{N_c} m)$  terms.

## $m_d - m_u$ effects in quark condensate

Current quark mass become diagonal  $2 \times 2$  matrix with  $m_1 = m_u, m_2 = m_d$ ,  $m = m_1 \frac{1+\tau_3}{2} + m_2 \frac{1-\tau_3}{2} = m + \delta m \frac{\tau_3}{2}$ . Here  $m = \frac{m_1+m_2}{2}$ ,  $\delta m = m_1 - m_2$ . Let us introduce external field  $s_i$ . In our particular case it is  $s_3 = i \frac{\delta m}{2}$ ,  $s_1 = s_2 = 0$ . Our aim is to find the asymmetry of the quark condensate  $\frac{\langle \bar{u}u \rangle - \langle \bar{d}d \rangle}{\langle \bar{u}u \rangle}$ , taking into account only  $O(\delta m)$  terms and neglecting by  $O(\frac{1}{N_c} \delta m)$ ,  $O(\frac{1}{N_c} \delta m \ln m)$ . It means that we neglect at all by meson loops contribution.

In the presence of the external field  $\vec{s}$  we expect also vacuum field  $\vec{\sigma}$ . Effective potential within requested accuracy is

$$V_{eff}[\sigma, \vec{\sigma}, m] \approx S_0[m, \lambda, \sigma, \vec{\sigma}] \\ = \frac{V}{2} (\sigma^2 + \vec{\sigma}^2) - \text{Tr} \ln \frac{\hat{p} + i\vec{\tau}\vec{s} + i(m + M(\lambda, \sigma, \vec{\sigma})F^2)}{\hat{p} + im + i\vec{\tau}\vec{s}} - N \ln \frac{K}{\lambda} + N. \quad (46)$$

$\lambda, \sigma, \vec{\sigma}$  are defined by the vacuum equations:

$$\frac{\partial V_{eff}}{\partial \lambda} = 0, \quad \frac{\partial V_{eff}}{\partial \sigma} = 0, \quad \frac{\partial V_{eff}}{\partial \sigma_i} = 0. \quad (47)$$

They can be reduced to the following form:

$$\frac{1}{2} \text{Tr} \frac{F^2(p) M_i (m_i + M_i F^2(p))}{p^2 + (m_i + M_i F^2(p))^2} = N \quad (48)$$

where  $M_i = \frac{\lambda^{0.5}}{2g} (2\pi\rho)^2 (\sigma \pm \sigma_3)$ . Solution of these equations leads to  $\lambda = \lambda[m, \vec{s}]$ ,  $\sigma = \sigma[m, \vec{s}]$   $\sigma_i = \sigma_i[m, \vec{s}]$ . We have to put them into  $V_{eff}$  and find  $V_{eff} = V_{eff}[m, \vec{s}]$ . Desired correlator is

$$\left. \frac{\partial V_{eff}[m, \vec{s}]}{\partial s_3} \right|_{s_3 = \frac{\delta m}{2}, s_{1,2}=0} \quad (49)$$

We calculate this correlator within requested accuracy, taking into account only  $O(\delta m)$  terms. So, the difference of the vacuum quark condensates of  $u$  and  $d$  quarks is

$$\begin{aligned} & \langle \bar{u}u \rangle - \langle \bar{d}d \rangle \\ &= \frac{1}{V} \left[ \text{Tr} \left( \frac{-i}{\hat{p} + i(m_u + M_u F^2)} - \frac{-i}{\hat{p} + im_u} \right) - \text{Tr} \left( \frac{-i}{\hat{p} + i(m_d + M_d F^2)} - \frac{-i}{\hat{p} + im_d} \right) \right] \end{aligned} \quad (50)$$

We expect that  $\langle \bar{d}d \rangle < \langle \bar{u}u \rangle$  if  $m_d > m_u$ .

Typical values of light current quark masses [19] are  $m_u = 5.1 \text{ MeV}$ ,  $m_d = 9.3 \text{ MeV}$  on the scale  $1 \text{ GeV}$  (which is in fact close to our scale  $\rho^{-1} = 0.6 \text{ GeV}$ ) leads to the asymmetry

$$\frac{\langle \bar{u}u \rangle - \langle \bar{d}d \rangle}{\langle \bar{u}u \rangle} = 0.026 \quad (51)$$

From this asymmetry and using sum-rules [20] we estimate strange quark condensate at  $m_s = 120 \text{ MeV}$  as:

$$\frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle} = 0.43, \quad (52)$$

which is rather small. The reason that the asymmetry (51) is rather large.

## Conclusion

In the framework of instanton vacuum model it was calculated simplest possible correlator – quark condensate with complete account of  $O(m)$ ,  $O(\frac{1}{N_c})$ ,  $O(\frac{1}{N_c}m)$  and  $O(\frac{1}{N_c}m \ln m)$  terms, demanding the calculation of meson loops contribution. Since initial instanton generated quark-quark interactions are nonlocal and contain corresponding form-factor induced by quark zero-mode, these loops correspond completely convergent integrals. The main loop corrections come from the pions, as it was expected. We found that  $O(\frac{1}{N_c})$  corrections are very large  $\sim 50\%$ , which request the  $\sim 10\%$  changing of the basic parameters – average inter-instanton distance  $R$  and average instanton size  $\rho$  to restore chiral limit value of the quark condensate  $\langle \bar{q}q \rangle_{m=0}$  and other important quantities as  $f_\pi$  and  $m_\pi$  to their phenomenological values. This work in the progress.

In general, it was demonstrated, that instanton vacuum model is well working tool also beyond chiral limit and satisfy chiral perturbation theory.

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